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Modelling of the Motion of a Disk Rolling on a Smooth Rigid Surface

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Abstract—This letter deals with the modelling of the motion of a disk rolling without slipping on a rigid surface.

Keywords—Rolling disk, Rigid surface, Nonholonomic constraints.

INTRODUCTION

This letter deals with the modelling of the motion of a disk rolling without slipping on a rigid surface. In this work, the fundamental building blocks for the dynamical model of the motion are derived for a class of surfaces. That is, the mathematical expressions for the disk's angular velocity vector, the disk's center of mass velocity vector, and the nonholonomic constraints at the point of contact between the disk and the surface are derived for a class of surfaces. Using these results, one can construct, for a given member of this class of surfaces, the Lagrangian function [1], and consequently, derive the Lagrange equations for the disk's motion.

DYNAMICAL MODEL

This work deals with the modelling of the motion of a disk rolling without slipping on a rigid surface, given below. Let $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ be unit vectors along an inertial (X, Y, Z)-coordinate system. It is assumed here that the surface is represented by

$$\mathbf{r}_S(u,v) = x(u,v)\mathbf{I} + y(u,v)\mathbf{J} + z(u,v)\mathbf{K}, \qquad (u,v) \in \mathcal{D}, \tag{1}$$

where $\mathcal{D}$ is an open set in $\Re^2$. It is assumed here that the mapping $\mathbf{r}_S$ of $\mathcal{D}$ into $\Re^3$ is a *coordinate patch of class* C^m $(m \geq 1)$. That is,

(i) $\mathbf{r}_S$ is of class C^m on $\mathcal{D}$;
(ii) $\frac{\partial \mathbf{r}_S}{\partial u} \times \frac{\partial \mathbf{r}_S}{\partial v} \neq 0$ for all $(u,v) \in \mathcal{D}$;
(iii) $\mathbf{r}_S$ is **1-1** and bicontinuous on $\mathcal{D}$ (see [2] for more details).

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It is further assumed here that: the shape of the terrain (given by (1)), and the radius a of the disk are such that during the disk's motion, at each instant, there is only one point of contact between the disk and the terrain.

Define the following vectors:

$$\mathbf{e}_1 = \left\| \frac{\partial \mathbf{r}_S}{\partial u} \right\|^{-1} \frac{\partial \mathbf{r}_S}{\partial u}, \qquad \mathbf{e}_2 = \left\| \frac{\partial \mathbf{r}_S}{\partial v} \right\|^{-1} \frac{\partial \mathbf{r}_S}{\partial v}, \qquad \mathbf{e}_3 = \left\| \frac{\partial \mathbf{r}_S}{\partial u} \times \frac{\partial \mathbf{r}_S}{\partial v} \right\|^{-1} \left(\frac{\partial \mathbf{r}_S}{\partial u} \times \frac{\partial \mathbf{r}_S}{\partial v} \right). \tag{2}$$

It is assumed here that *the vectors* $\mathbf{e}_k$, $k = 1,2,3$ *constitute an orthonormal system*. Thus, this work is confined to a class of surfaces for which this assumption holds. Hence, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal coordinate system, $\mathbf{e}_k = \mathbf{e}_k(u, v)$, $k = 1,2,3$. Let $\boldsymbol{\Omega} = \Omega_1 \mathbf{e}_1 + \Omega_2 \mathbf{e}_2 + \Omega_3 \mathbf{e}_3$ denote the vector angular velocity of $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, then, from the relations

$$\frac{d\mathbf{e}_k}{dt} = \boldsymbol{\Omega} \times \mathbf{e}_k, \qquad k = 1,2,3,$$

and equations (1),(2), it follows that:

$$\frac{d\mathbf{e}_1}{dt} = \Omega_3 \mathbf{e}_2 - \Omega_2 \mathbf{e}_3, \qquad \frac{d\mathbf{e}_2}{dt} = -\Omega_3 \mathbf{e}_1 + \Omega_1 \mathbf{e}_3, \qquad \frac{d\mathbf{e}_3}{dt} = \Omega_2 \mathbf{e}_1 - \Omega_1 \mathbf{e}_2, \tag{3}$$

where $\Omega_k = \Omega_k(u, \frac{du}{dt}, v, \frac{dv}{dt})$, $k = 1,2,3$. It is assumed here that the origin of $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is attached to, and moving with, the point of contact $\mathbf{r}_C$ between the disk and the surface. Note that

$$\mathbf{r}_C = x(u_C, v_C)\mathbf{I} + y(u_C, v_C)\mathbf{J} + z(u_C, v_C)\mathbf{K}, \tag{4}$$

where $(u_C, v_C) \in \mathcal{D}$. Denote by $\mathbf{k}$

$$\mathbf{k} = \sin\theta \cos\phi \mathbf{e}_1 + \sin\theta \sin\phi \mathbf{e}_2 + \cos\theta \mathbf{e}_3, \tag{5}$$

a unit vector along the axis of the disk. Then, the vectors $\mathbf{i}_\theta$ and $\mathbf{i}_\phi$, given by $\mathbf{i}_\theta = \frac{\partial \mathbf{k}}{\partial \theta}$ and $\mathbf{i}_\phi = (1/\sin\theta)\frac{\partial \mathbf{k}}{\partial \phi}$ are at all times in the plane of the disk. Define the following vectors, $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{i} = \cos\psi \mathbf{i}_\theta + \sin\psi \mathbf{i}_\phi, \qquad \mathbf{j} = -\sin\psi \mathbf{i}_\theta + \cos\psi \mathbf{i}_\phi, \tag{6}$$

which are always in the plane of the disk. Hence, the following relations are obtained:

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \mathbf{E}(\theta, \phi, \psi) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{7}$$

where

$$\mathbf{E} = \begin{pmatrix} \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi & \cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi & -\sin\theta\cos\psi \\ -\cos\theta\cos\phi\sin\psi - \sin\phi\cos\psi & -\cos\theta\sin\phi\sin\psi + \cos\phi\cos\psi & \sin\theta\sin\psi \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}.$$

For such a relation between $(\mathbf{i}, \mathbf{j}, \mathbf{k})^\top$ and $(\mathbf{I}, \mathbf{J}, \mathbf{K})^\top$ see, for example, [1]. By using the relations

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega}_D \times \mathbf{i}, \quad \frac{d\mathbf{j}}{dt} = \boldsymbol{\omega}_D \times \mathbf{j}, \quad \frac{d\mathbf{k}}{dt} = \boldsymbol{\omega}_D \times \mathbf{k}, \qquad \boldsymbol{\omega}_D = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}, \tag{8}$$

where $\boldsymbol{\omega}_D$ is the angular velocity vector of the disk, the following equations are obtained:

$$\omega_1 = \frac{d\theta}{dt}\sin\psi - \frac{d\phi}{dt}\sin\theta\cos\psi + \sum_{k=1}^{3} e_{1k}\Omega_k, \qquad \omega_2 = \frac{d\theta}{dt}\cos\psi + \frac{d\phi}{dt}\sin\theta\sin\psi + \sum_{k=1}^{3} e_{2k}\Omega_k,$$

$$\omega_3 = \frac{d\psi}{dt} + \frac{d\phi}{dt}\cos\theta + \sum_{k=1}^{3} e_{3k}\Omega_k,$$

where the notation $\mathbf{E}_{ij} = e_{ij}$, $i, j = 1, 2, 3$, is being used as above. Also, since $\boldsymbol{\omega}_D = \omega_{e1}\mathbf{e}_1 + \omega_{e2}\mathbf{e}_2 + \omega_{e3}\mathbf{e}_3$, it follows that:

$$\omega_{e1} = \frac{d\psi}{dt}\sin\theta\cos\phi - \frac{d\theta}{dt}\sin\phi + \Omega_1, \qquad \omega_{e2} = \frac{d\psi}{dt}\sin\theta\sin\phi + \frac{d\theta}{dt}\cos\phi + \Omega_2,$$

$$\omega_{e3} = \frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta + \Omega_3,$$

where, in both representations of $\boldsymbol{\omega}_D$, that is, in the $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, representations, $\Omega_k = \Omega_k(u_C, \frac{du_C}{dt}, v_C, \frac{dv_C}{dt})$, $k = 1, 2, 3$. Let $\mathbf{r}_O$ denote the center of the disk, and denote by $\mathbf{r}_C$ the point of contact between the disk and the rigid terrain, thus,

$$\mathbf{r}_O = \mathbf{r}_C - a\mathbf{i}_\theta, \tag{9}$$

where a is the radius of the disk. It follows from (2) and (4)

$$\frac{d\mathbf{r}_C}{dt} = \alpha(\boldsymbol{\eta})\frac{du_C}{dt}\mathbf{e}_1 + \beta(\boldsymbol{\eta})\frac{dv_C}{dt}\mathbf{e}_2, \qquad \alpha(\boldsymbol{\eta}) = \left\|\frac{\partial\mathbf{r}_C(\boldsymbol{\eta})}{\partial u_C}\right\|, \quad \beta(\boldsymbol{\eta}) = \left\|\frac{\partial\mathbf{r}_C(\boldsymbol{\eta})}{\partial v_C}\right\|, \tag{10}$$

where $\boldsymbol{\eta} = (u_C, v_C)$ and $\mathbf{e}_k = \mathbf{e}_k(\boldsymbol{\eta})$, $k = 1, 2, 3$. Thus, by using (9), (10), and the expression for $\mathbf{i}_\theta$, equation (9) yields

$$\mathbf{v}_O = \frac{d\mathbf{r}_O}{dt} = v_{e1}\mathbf{e}_1 + v_{e2}\mathbf{e}_2 + v_{e3}\mathbf{e}_3, \tag{11}$$

where

$$v_{e1} = \alpha(\boldsymbol{\eta})\frac{du_C}{dt} + a\left(\frac{d\theta}{dt}\sin\theta\cos\phi + \frac{d\phi}{dt}\cos\theta\sin\phi\right) + a\Omega_2\sin\theta + a\Omega_3\cos\theta\sin\phi, \tag{12}$$

$$v_{e2} = \beta(\boldsymbol{\eta})\frac{dv_C}{dt} + a\left(\frac{d\theta}{dt}\sin\theta\sin\phi - \frac{d\phi}{dt}\cos\theta\cos\phi\right) - a\Omega_1\sin\theta - a\Omega_3\cos\theta\cos\phi, \tag{13}$$

$$v_{e3} = a\cos\theta\left(\frac{d\theta}{dt} - \Omega_1\sin\phi + \Omega_2\cos\phi\right). \tag{14}$$

Let $\mathbf{H}(u_C, v_C)$ be an orthonormal matrix such that $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^\top = \mathbf{H}(u_C, v_C)(\mathbf{I}, \mathbf{J}, \mathbf{K})^\top$. Then, one can use (4) and (9) to obtain $V = m_D g\langle(\mathbf{r}_C - a\mathbf{i}_\theta), \mathbf{K}\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the inner product between two vectors, and m_D denotes the mass of the disk. The mathematical expression of $\mathbf{H}$ depends on the form of the surface $\mathbf{r}_S$.

Denote by $I_{Dj}, j = 1, 2, 3$, the moments of inertia of the disk about the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ axes, respectively. Using the "thin wheel" approximation, we have $I_{D1} = I_{D2} = 0.25m_D a^2$, $I_{D3} = 0.5m_D a^2$.

Thus, the Lagrangian function for the system's motion is given by

$$\mathcal{L} = \frac{1}{2}I_{D1}\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}I_{D3}\omega_3^2 + \frac{1}{2}m_D v_O^2 - V, \tag{15}$$

where V is given above and $v_O^2 = \|\mathbf{v}_O\|^2$. Thus, θ, ϕ, and ψ are playing here the following roles: θ is the leaning angle of the disk, that is, the angle between the disk's axis and the $\mathbf{e}_3$-axis where for $\theta = \pi/2$ the plane of the disk is vertical to the $(\mathbf{e}_1, \mathbf{e}_2)$-tangent plane (at the point $\mathbf{r}_C$); $\mathbf{i}_\phi$ represents the direction of the disk on the $(\mathbf{e}_1, \mathbf{e}_2)$-tangent plane to the surface at $\mathbf{r}_C$; ψ represents the angle of rotation of the disk around its axis; and (u_C, v_C) represents the point of contact on the surface.

The motion of the disk involves rolling without slipping. This leads to the condition (see, for example, [1])

$$\mathbf{v}_O + \boldsymbol{\omega}_D \times a\mathbf{i}_\theta = \mathbf{0}, \tag{16}$$

at $\mathbf{r}_C$, where $\mathbf{v}_O$ and $\boldsymbol{\omega}_D$ are given in the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ basis. Hence, equation (16) leads to

$$\alpha(\eta)\frac{du_C}{dt} - a\frac{d\psi}{dt}\sin\phi = 0, \qquad \beta(\eta)\frac{dv_C}{dt} + a\frac{d\psi}{dt}\cos\phi = 0. \tag{17}$$

Denote

$$\mathbf{A} = \begin{pmatrix} -a\sin\phi & \alpha(\eta) & 0 \\ a\cos\phi & 0 & \beta(\eta) \end{pmatrix}. \tag{18}$$

Since rank $\mathbf{A} = 2$ on $\mathcal{D}$, it follows that equations (17) constitute two independent nonholonomic constraints for the motion of the disk on the surface given by (1).

Hence, having the expressions for ω_k, v_{ek}, $k = 1, 2, 3$, V, and equations (17), one can derive the equations of motion of the disk by using the Lagrangian, (15), and equations (17).

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